September 2015 Puzzle

Obtain a nine-digit number consisting of each of the digits 1 – 9 (each exactly once) so that the nine conditions below are satisfied: The number is divisible by 9; The number obtained by omitting the rightmost digit in this number is divisible by 8; The number obtained by omitting the two rightmost digits in this number is divisible by 7; ... The number obtained by omitting the eight rightmost digits in this number is divisible by 1.

September 2015 Solution

The answer is: 381654729

To confirm this solution, observe that:

\[
\begin{align*}
3/1 &= 3; \\
38/2 &= 19; \\
381/3 &= 127; \\
3816/4 &= 954; \\
38165/5 &= 7633; \\
381654/6 &= 63609; \\
3816547/7 &= 545221; \\
38165472/8 &= 4770684; \\
381654729/9 &= 42406081
\end{align*}
\]

We will now demonstrate that the solution provided above is unique. Let us represent a number satisfying the conditions of this problem by \(abcdefghi\), where \(a \ldots i\) represent the 9 digits of the number. Thus, the nine conditions above can be reformulated as: \(abcdefghi\) is divisible by 9, \(abcdefgh\) is divisible by 8, \(abcdefg\) is divisible by 7, and so on. Throughout the explanation below, we will be making use of “well known” divisibility rules for number, some of which you will be familiar with and perhaps some of which you are not.

We will label the steps of our argument using capital letters, A, B, C, etc...

A. It is first worth noting that the conditions that the number is divisible by 9 and that the number obtained by omitting the eight rightmost digits is divisible by 1 will be satisfied by any permutation of the digits 1 – 9 and thus, those conditions can be ignored.

B. We are told that \(ab\) is divisible by 2, \(abcd\) is divisible by 4, \(abcdef\) is divisible by 6, and \(abcdefgh\) is divisible by 8. Any number that is divisible by an even number must end in an even digit and therefore we know that \(b, d, f,\) and \(h\) are even, which then requires \(a, c, e, g,\) and \(i\) to be odd.

C. Any number that is divisible by 5 must end in a 0 or a 5. Since \(abcde\) is divisible by 5, \(e\) must be either 0 or 5. However, from B we know that \(e\) is odd and therefore conclude that \(e = 5\).

D. Any number that is divisible by 4 must have its last two (rightmost) digits divisible by 4. Since \(abcd\) is divisible by 4, we must have \(cd\) divisible by 4. But we know that \(c\) is odd, so this only leaves the following possibilities for \(cd\): 12, 16, 32, 36, 52, 56, 72, 76, 92, or 96. This means that \(d = 2\) or \(6\).

E. Now, \(abcdefgh\) is divisible by 8, which means it is also divisible by 4. Applying the same argument given in D, we can conclude that \(h = 2\) or \(6\).

F. Now that we know \(d = 2\) or \(6\) and \(h = 2\) or \(6\), the only possibilities left for \(f\) are \(f = 4\) or \(8\).

G. Since \(abcdef\) is divisible by 6, it is also divisible by 3. Additionally, we know that \(abc\) is divisible by 3. For a number to be divisible by 3, the sum of its digits must also be divisible by 3. This means that both \(a + b + c + d + e + f\) and \(a + b + c\) are divisible by 3. This then requires that \(d + e + f\) be divisible by 3, which then implies that \(def\) is divisible by 3 as well. From C we know that \(e = 5\) and from F we know that \(d = 2\) or 6 and \(f = 4\) or 8. The only possibilities for \(def\) resulting in a number that is divisible by 3 is \(def = 258\) or 654.

H. As noted above, the fact that \(abc\) is divisible by 3 implies that \(a + b + c\) is divisible by 3. Recall that \(a, b,\) and \(c\) are distinct digits with \(a\) and \(c\) odd and \(b\) even, resulting in \(a + b + c\) being even as well. One can therefore determine that the only possibilities for \(a + b + c\) are 24, 18, or 12.

To obtain \(a + b + c = 24\), we would need \(abc = 987\) or 789.
To obtain \(a + b + c = 18\), we would need \(abc = 783, 387, 981,\) or 189.
To obtain \(a + b + c = 12\), we would need \(abc = 381, 183, 741,\) or 147.

I. We now return to the fact that \(def = 258\) or 654 which we established in G. If \(def = 258\), then because each of the digits are distinct, the only possibility for \(abc\) from those mentioned above are \(abc = 741\) or 147. With \(d = 2\) and the fact that \(h = 2\) or 6 (see E), we would have to have \(h = 6\). This would give us the number 741258g6i or 147258g6i, leaving 3 and 9 to take the places of \(g\) and \(i\). One can easily check that neither of these options results in a number satisfying all of the desired conditions. We therefore conclude that \(def = 258\) is impossible, so we must have \(def = 654\).

J. With \(def = 654\) we can conclude that \(h = 2\), since the only possibilities for \(h\) were 2 or 6 (see E) and we now know \(d = 6\). Additionally, the fact that \(f = 4\) eliminates the possibilities of \(abc = 741\) or 147 as given in H. All of the remaining possibilities for \(abc\) require that \(b\) is equal to 8 and thus we conclude that \(b = 8\).

K. We are given that \(abcdefgh\) is divisible by 8, which as noted earlier requires the last three digits, \(fgh\) to be divisible by 8. But \(fgh = 4g2\), and \(g\) is odd leaving only the possibilities, \(g = 3\) or 7. This eliminates the possibilities that \(abc = 783\) or 387.

L. We know that the entire number \(abcdefghi\) is divisible by 9 and therefore divisible by 3 as well. Because it is divisible by 3, we know that \(a + b + c + d + e + f + g + h + i\) is divisible by 3 and we have already established in G that \(a + b + c + d + e + f\) is divisible by 3. From this we conclude that \(g + h + i\) is divisible by 3 and therefore \(ghi\) is divisible by 3 as well. From K we know \(g = 3\) or 7 and from J, we know \(h = 2\). We will now show that \(g\) can’t be 3. If \(g = 3\), then \(ghi = 32i\). The only remaining possibilities for \(i\) would be 1 or 7 which we will rule out below using the results of H and the fact that we eliminated the possibilities that \(abc = 741, 147, 783, \) or 387 (see J and K).

If \(i = 1\), then the only possibilities left for \(abc\) would be 987 or 789. These cases would result in the number being 987654321 or 789654321, neither of which satisfy all of the conditions. So we can’t have \(i = 1\).
If \( i = 7 \), then the only possibilities left for \( abc \) would be 981 or 189. These cases would result in the number being 981654327 or 189654327, neither of which satisfy all of the conditions. So we can’t have \( i = 7 \) either.

Since \( i = 1 \) or 7 were the only possibilities in the case that \( g = 3 \) and we have shown the neither lead to a solution, we conclude that \( g \neq 3 \) leaving us with \( g = 7 \) as the only remaining possibility.

M. Now that we know \( g = 7 \) and \( h = 2 \), we have \( ghi = 72i \). Recall from N that \( ghi \) is divisible by 3 and \( i \) is odd. From the numbers that remain, this only allows for the possibilities that \( i = 3 \) or 9. Again we use the results of H, J, and K to rule out the possibility of \( i = 3 \) as follows.

If \( i = 3 \), then the only remaining possibilities for \( abc \) would be 981 or 189, neither of which would lead to a number satisfying all of the conditions.

We thus conclude that \( i = 9 \), which only leaves the possibilities of 381 and 183 for \( abc \). One can easily check that 183 does not lead to a number satisfying all of the conditions, but 381 does. With \( abc = 381 \), we have obtained our unique solution

\[
\text{abcdefghi} = 381654729
\]