Three men are racing up a set of stairs. The image to the right shows the final moments before the end of the race. Up until the final leaps needed to reach the top: the man in the lead leapt three steps at a time and is one step from the top; the man in second place leapt four steps at a time and is seven steps from the top; and the man in third place leapt five steps at a time and is fourteen steps from the top (as you might imagine, these are rather short stairs). Given that each of the men began at the bottom and that there were at least 100 steps total, how many steps were there? The top landing should be counted as a step, but not the bottom. Also determine the number of leaps taken by each of the men in completing the race.

Mathematically speaking, there are infinitely many solutions to this puzzle. The answers for the solution involving the least number of steps are as follows.

There are 139 steps.
The man in the lead took 47 leaps to complete the race.
The man in second place took 35 leaps to complete the race.
The man in third place took 28 leaps to complete the race.

To arrive at these results we let:

\[ N = \text{total number of stairs}; \]
\[ a = \text{the number of leaps of “size 3” taken by the man in 1st, at the point shown in the figure}; \]
\[ b = \text{the number of leaps of “size 4” taken by the man in 2nd, at the point shown in the figure}; \]
\[ c = \text{the number of leaps of “size 5” taken by the man in 3rd, at the point shown in the figure}. \]

Based on the number of stairs remaining at the point shown in the figure, we must have

\[ N = 3a + 1, \quad N = 4b + 7 \quad \text{and} \quad N = 5c + 14 \]

Setting the last two expressions for \( N \) equal to one another, we obtain:

\[ 4b + 7 = 5c + 14 \quad \rightarrow \quad 4b + 7 = c + 4c + 14 \quad \rightarrow \quad c = 4b - 4c - 7 \quad \rightarrow \quad c = 4(b - c - 2) + 1 \]
If we let \( k = b - c - 2 \) we can simply write \( c = 4k + 1 \). Inserting this back into the third expression for \( N \) gives us

\[
N = 5c + 14 = 5(4k + 1) + 14 = 20k + 19
\]

Setting this expression for \( N \) equal to the first expression for \( N \) given on the previous page, we obtain

\[
20k + 19 = 3a + 1 \quad \rightarrow \quad 2k + 18k + 19 = 3a + 1 \quad \rightarrow \quad 2k = 3a - 18k - 18
\]

\[
\rightarrow \quad 2k = 3(a - 6k - 6)
\]

If we let \( m = a - 6k - 6 \) we have \( 2k = 3m \) or simply \( k = \frac{3m}{2} \).

Since \( k \) is an integer (recall \( k = b - c - 2 \)), it turns out that \( m \) has to contain a factor of 2 and if we let \( m = 2n \), then we have \( k = \frac{3(2n)}{2} = 3n \). Inserting this back into the expression \( 20k + 19 \), we end up with

\[
N = 20(3n) + 19 \quad \text{or simply} \quad N = 60n + 19
\]

So what we have concluded, is that for \( N \) to satisfy all three conditions given on the first page, \( N \) must take the form \( N = 60n + 19 \) where \( n \) is an integer. Furthermore, as demonstrated below, if \( N = 60n + 19 \) where \( n \) is any integer, then \( N \) will satisfy all three conditions.

\[
60n + 19 = 3\left(\frac{20n + 6}{a}\right) + 1; \quad 60n + 19 = 4\left(\frac{15n + 3}{b}\right) + 7; \quad 60n + 19 = 5\left(\frac{12n + 1}{c}\right) + 14
\]

Since the number of stairs must be at least 100, we see that the smallest value of \( n \) yielding a solution is \( n = 2 \) which gives us \( N = 60(2) + 19 = 139 \) stairs. However, any larger integer value of \( n \) would yield another possible number of stairs that would satisfy the given conditions (though there is certainly a practical limit to how many stairs could exist in a set of stairs that men could race up).

Working with the solution of \( N = 139 \) stairs, we can solve each of the three conditions on the previous page for \( a, b, \) and \( c \) to obtain \( a = 46, b = 33, \) and \( c = 25 \). However, these values indicate the number of leaps taken by each of the corresponding men to arrive at the positions shown in the figure. The man in 1\(^{\text{st}}\) place must take one additional leap to finish the race; the man in 2\(^{\text{nd}}\) place must take two additional leaps to finish the race; and the man in 3\(^{\text{rd}}\) place must take three additional leaps to finish the race.

Thus we conclude that the man in first required 47 leaps to finish the race; the man in 2\(^{\text{nd}}\) required 35 leaps to finish the race; and the man in 3\(^{\text{rd}}\) required 28 leaps to finish the race.